What is pure mathematics all about?

experienced mathematicians generally agree on which concepts are more important than others

(deeper justification than physics)

Hope to discuss:

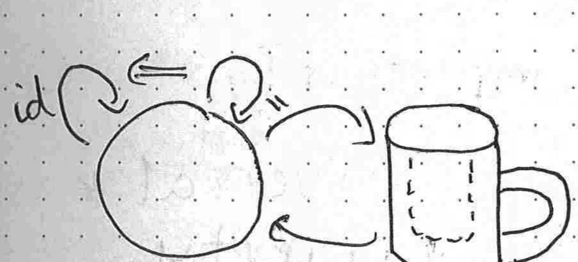
- homotopification: invading math, (mug – donut – circle?)

via <u>n-category theory</u>

(2d-surface) $\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$ (combinatorial, not topological)

"up to equivalence" $\mathrm{Vect}_{\mathbb{R}}^{fd} \simeq \{\mathbb{R}, \mathbb{R}^2, \ldots\}$

$\Rightarrow$ linear maps $\simeq$ matrices



- ✶ Homotopy Type Theory ✶

and

- homological algebra

- classification of groups
  - interesting, hard, pertinent problem

finite abelian groups
– nontrivial, interesting

finite groups
– HARD

or more generally, classifying
⊙ homotopy types

⊙ simplicial methods 

— very practical, computable,
beautiful

---

certain structures are particularly important:

$$\begin{tikzcd} \mathbb{N} \arrow[r, hook] & \mathbb{Z} \arrow[r, hook] & \mathbb{Q} \arrow[r, hook] \arrow[d, hook] & \mathbb{R} \arrow[r, hook] & \mathbb{C} \\ & & \mathbb{Q}_p & & \end{tikzcd}$$

$\mathbb{Q}_p$ p-adics

why? one reason: "freedom"
— $\mathbb{Z}$ is the <u>free group</u> on one generator, 1.
naïvely, just knowing $1 \in \mathbb{Z}$, also contains
$\ldots, -1+-1, -1, 0, 1, 1+1, \ldots$
(don't throw in more stuff or properties
than I told you too)
how does that make it important?

(mixing maps of different kinds)

$$\begin{tikzcd} \{1\} \arrow[r, "\iota"] \arrow[dr, "\forall f"'] & \mathbb{Z} \arrow[d, "\exists ! \tilde{f}"] \\ & G \end{tikzcd}$$

universal * property

freeness $\Rightarrow$ $\tilde{f}$ is <u>determined</u> by $\tilde{f}\iota(1)$.
(all homomorphisms $\mathbb{Z} \to G$ are determined
by any $g \in G$.)

$\mathbb{Z}$ is the "walking element" in Grp.

Dolan — "that guy is just a walking pair of eyebrows"

last diagram: two of the morphisms were in Set = [sets, functions]

and one was a morphism in Grp = [groups, homomorphisms]

ie we're exploiting a functor

$$U : \mathrm{Grp} \to \mathrm{Set}$$

really, there is a natural isomorphism

$$\star \quad \mathrm{Grp}(\mathbb{Z}, G) \overset{\alpha}{\cong} \underbrace{\mathrm{Set}(\{1\}, UG)}_{\text{(elements of } G\text{)}}$$

so in our triangle, $\tilde{f} = \alpha^{-1}(f)$ is the unique homo that made $\triangledown$ commute.

why $\star$ equivalent to that commutativity?

<u>naturality</u>

HW: ① starting with $\star$ (including naturality of α) where does $\iota : \{1\} \to U\mathbb{Z}$ come from?

② why does the triangle commute?

$$\begin{tikzcd}
\{1\} \arrow[r, "\iota"] \arrow[dr, "f"'] & U\mathbb{Z} \arrow[d, "U\alpha^{-1}(f)"] \\
& UG
\end{tikzcd}$$