10/1/18 (Seminar 2)

to catch up with basic definitions,
taught a course a couple years ago:
http://math.ucr.edu/home/baez/qg-winter2016/

- continuing: freedom
  function as "probes" $(\mathbb{Z} \to G)$

$\mathbb{Z} = F(\{1\}) \Rightarrow$ ① $\mathbb{Z}$ is abelian,
because 1 commutes with itself

moreover, $\mathbb{Z}$ is the free abelian group

$$(\mathrm{Set} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathrm{Ab})$$

$$\alpha_A : \mathrm{Ab}(\mathbb{Z}, A) \cong \mathrm{Set}(1, U(A))$$

but also,

② $\mathbb{Z}$ is a <u>ring</u>: by freeness,

$$\forall n \in \mathbb{Z} \; \exists!\, n \cdot - : \begin{matrix} \mathbb{Z} \to \mathbb{Z} \\ 1 \mapsto n \end{matrix}$$ (multiplication

since it's a homomorphism,
we get the <u>distributive law</u>

$$n \cdot (a+b) = na + nb$$
$$n \cdot 0 = 0$$

(exercise: prove multiplication is associative)

③ $\mathbb{Z}$ is the <u>free</u> ring on <u>zero</u> generators

$$\alpha_R : \mathrm{Ring}(\mathbb{Z}, R) \cong \mathrm{Set}(\emptyset, U(R))$$

$\Rightarrow \exists!$ ring homomorphism $\mathbb{Z} \to R$ $\forall R$: Ring
(ie, $\mathbb{Z}$ is the initial object in Ring)

(4) in fact, $\mathbb{Z}$ is a commutative ring
(puzzle: show this starting from #3)

(and the free commutative ring) *

— we don't only care about negatives

"big rig" if proper class

$\mathbb{N}$ is a monoid, but also a rig
(ring without negatives)

monoids $\supset$ groups
rigs $\supset$ rings

(0) $\mathbb{N}$ is the free monoid on one generator
(1) $\mathbb{N}$ is the free commutative monoid "
(2) $\mathbb{N}$ is the free rig on no generators
(3) $\mathbb{N}$ is the free commutative rig "

ring has both underlying (additive) group
and " (multiplicative) monoid

(a rig $(R, +, 0, \cdot, 1)$ has
two underlying monoids
$(R, +, 0)$ and $(R, \cdot, 1)$)

<u>Thm</u>: the monoid $(\mathbb{N}^*, \cdot, 1)$ is the free commutative monoid on a countable number of generators

can't have 0 to be free

<u>Prf</u>: they are $\{2, 3, 5, 7, \ldots\}$ (primes)

- every natural number is a product of primes in a unique way, up to mult by 1, assoc, commutative (Fundamental Thm of Arithmetic)

free + underlying functors

$$\alpha_G : \mathrm{Grp}(F(S), G) \cong \mathrm{Set}(S, U(G))$$

(F is a functor – a function $f: S \to S'$ gives a homo. $Ff: FS \to FS'$)

- check functoriality

(more structure)

"free" left adjoint

$$\begin{tikzcd} \mathrm{Grp} \arrow[d, bend left, "U"] \arrow[d, phantom, "\dashv"] \\ \mathrm{Set} \arrow[u, bend left, "F"] \end{tikzcd}$$

"forgetful" right adjoint (less structure)

liberal

conservative

$$\begin{tikzcd} D \arrow[d, bend left, "U"] \\ C \arrow[u, bend left, "F"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

F is <u>left adjoint</u> of U or equivalently U is <u>right adjoint</u> of F if there is a natural iso

$$\alpha_{cd} : D(Fc, d) \cong C(c, Ud)$$

(functors $C^{op} \times D \to Set$ – natural in <u>both</u> variables)

Q & A: hw question #2

use commuting $\square$ to get commuting $\triangledown$

$$\begin{tikzcd}
1 \in Grp(\mathbb{Z},\mathbb{Z}) \arrow[r, leftrightarrow, "\alpha_{\mathbb{Z}}", "\sim"'] \arrow[d, "\tilde{f} \circ -"'] & Set(\{1\}, U(\mathbb{Z})) \ni i \arrow[d, "U(\tilde{f}) \circ -"] \\
Grp(\mathbb{Z}, G) \arrow[r, leftrightarrow, "\sim", "\alpha_G"'] & Set(\{1\}, U(G)) \\
\tilde{f} \arrow[r, leftrightarrow] & f
\end{tikzcd}$$

$$\Rightarrow \quad f = U(\tilde{f}) \circ i$$

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- check functoriality

(more structure)

$$\begin{tikzcd} \mathrm{Grp} \arrow[d, bend left=60, "U"] \arrow[d, phantom, "\dashv"] \\ \mathrm{Set} \arrow[u, bend left=60, "F"] \end{tikzcd}$$

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$$
\begin{tikzcd}
1 \arrow[r, phantom, "\in"] \arrow[ddr, bend right] \arrow[rrrr, bend left, leftrightarrow] & Grp(\mathbb{Z},\mathbb{Z}) \arrow[rr, leftrightarrow, "\alpha_{\mathbb{Z}}", "\sim"'] \arrow[d, "\tilde{f}\circ -"'] & & Set(\{1\}, U(\mathbb{Z})) \arrow[r, phantom, "\ni"] \arrow[d, "U(\tilde{f})\circ -"] & i \arrow[ddl, bend left] \\
& Grp(\mathbb{Z}, G) \arrow[rr, leftrightarrow, "\sim", "\alpha_G"'] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & Set(\{1\}, U(G)) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \\
& \tilde{f} \arrow[rr, leftrightarrow] & & f &
\end{tikzcd}
$$

$\Longrightarrow \quad f = U(\tilde{f}) \circ i$