

10/1/18 (Seminar 2)

to catch up with basic definitions,  
taught a course a couple years ago:

<http://math.ucr.edu/home/baez/99-winter2006/>

• continuing: freedom

function as "probes" ( $\mathbb{Z} \rightarrow G$ )

$\mathbb{Z} = F(\{1\}) \Rightarrow$  ①  $\mathbb{Z}$  is abelian,  
because 1 commutes with itself

moreover,  $\mathbb{Z}$  is the free abelian group

( $\text{Set} \xrightleftharpoons{\perp} \text{Ab}$ )

$\alpha_A: \text{Ab}(\mathbb{Z}, A) \cong \text{Set}(1, U(A))$

but also,

②  $\mathbb{Z}$  is a ring: by freeness,

$\forall n \in \mathbb{Z} \exists! n \cdot - : \mathbb{Z} \rightarrow \mathbb{Z}$  (multiplication)  
 $1 \mapsto n$

since it's a homomorphism,  
we get the distributive law

$$n \cdot (a+b) = na + nb$$

$$n \cdot 0 = 0$$

(exercise: prove multiplication is associative)

③  $\mathbb{Z}$  is the free ring on zero generators

$\alpha_R: \text{Ring}(\mathbb{Z}, R) \cong \text{Set}(\emptyset, U(R))$

$\Rightarrow \exists!$  ring homomorphism  $\mathbb{Z} \rightarrow R \quad \forall R: \text{Ring}$   
(ie,  $\mathbb{Z}$  is the initial object in Ring)

④ in fact,  $\mathbb{Z}$  is a commutative ring  
(puzzle: show this starting from #3)

(and the free commutative rings)\*

— we don't only care about negatives  
 $\mathbb{N}$  is a monoid, but also a rig "big rig" if proper class  
(ring without negatives)

monoids  $\supset$  groups  
rigs  $\supset$  rings

①  $\mathbb{N}$  is the free monoid on one generator

②  $\mathbb{N}$  is the free commutative monoid "

③  $\mathbb{N}$  is the free rig on no generators

④  $\mathbb{N}$  is the free commutative rig "

ring has both underlying (additive) group  
and " (multiplicative) monoid

(a rig  $(R, +, 0, \cdot, 1)$  has  
two underlying monoids  
 $(R, +, 0)$  and  $(R, \cdot, 1)$ )

Thm: the monoid  $(\mathbb{N}^*, \cdot, 1)$  can't have 0 to be free  
 is the free commutative monoid on a countable number of generators

Prf: they are  $\{2, 3, 5, 7, \dots\}$  (primes)

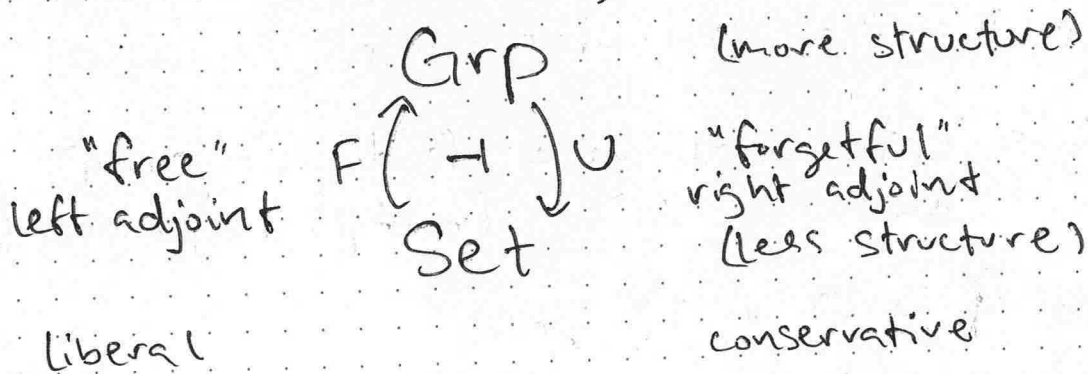
— every natural number is a product of primes in a unique way, up to mult by 1, assoc, commutative  
 (Fundamental Thm of Arithmetic)

free + underlying functors

$$\alpha_G: \text{Grp}(F(S), G) \cong \text{Set}(S, U(G))$$

(F is a functor — a function  $f: S \rightarrow S'$  gives a homo.  $Ff: FS \rightarrow FS'$ )

— check functoriality



$F \begin{pmatrix} \uparrow D \\ \dashv U \\ \downarrow C \end{pmatrix}$ 
 F is left adjoint of U or equivalently  
 U is right adjoint of F if there is a  
 natural iso

$$\alpha_{cd}: D(Fc, d) \cong C(c, Ud)$$

(functors  $C^{\text{op}} \times D \rightarrow \text{Set}$  - natural in both variables)

Q&A: hw question #2

use commuting  $\square$  to get commuting  $\nabla$

$$\begin{array}{ccc}
 1 \in \text{Grp}(\mathbb{Z}, \mathbb{Z}) & \xleftarrow[\sim]{\alpha_{\mathbb{Z}}} & \text{Set}(\{1\}, U(\mathbb{Z})) \ni i \\
 \tilde{f} \circ - \downarrow & & \downarrow U(\tilde{f}) \circ - \\
 \text{Grp}(\mathbb{Z}, G) & \xleftarrow[\sim]{\alpha_G} & \text{Set}(\{1\}, U(G)) \\
 \tilde{f} \circ - \downarrow & & \downarrow f \circ - \\
 & & f
 \end{array}$$

$$\Rightarrow f = U(\tilde{f}) \circ i$$

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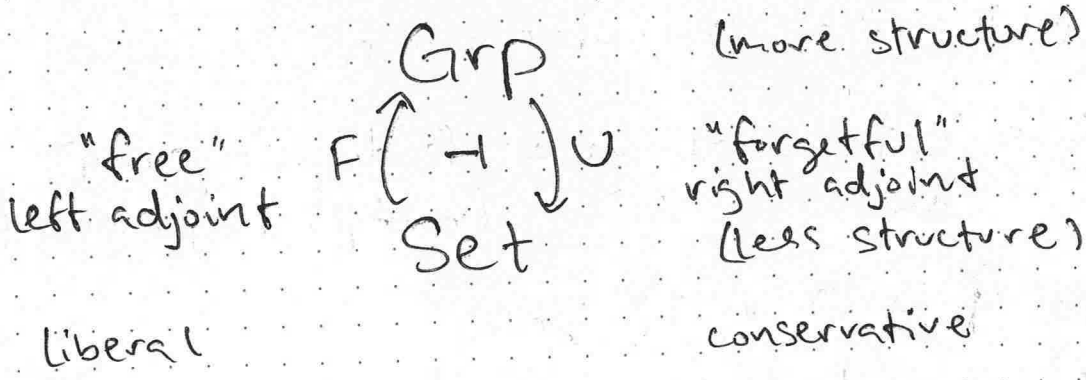
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 \downarrow \psi & & \downarrow \psi \\
 \tilde{f} & \xleftarrow{\quad} & f
 \end{array}$$

$$\Rightarrow f = U(\tilde{f}) \circ i$$