10/3 Christina Vasilakopoulous

(John is in UK today & Friday)

will need 2-categories for homotopy

$\mathcal{C}$ (locally small) category

$$\hom_{\mathcal{C}} = \mathcal{C}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Set}$$

$f : x' \to x$

$g : y \to y'$

$$\begin{tikzcd}
(x, y) \arrow[r, mapsto] \arrow[d, "{(f,g)}"'] & \mathcal{C}(x,y) \arrow[d, "g\circ - \circ f"] \\
(x', y') \arrow[r, mapsto] & \mathcal{C}(x',y')
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{D} \arrow[d, bend left, "U"] \arrow[d, phantom, "\dashv"] \\
\mathcal{C} \arrow[u, bend left, "F"]
\end{tikzcd}
\qquad \alpha_{c,d} : \mathcal{D}(Fc, d) \xrightarrow{\sim} \mathcal{C}(c, Ud)$$

$$\mathcal{C}^{op} \times \mathcal{D} \xrightarrow{F^{op} \times 1} \mathcal{D}^{op} \times \mathcal{D} \xrightarrow{\mathcal{D}(-,-)} \underline{\mathrm{Set}}$$

$$(c, d) \mapsto (Fc, d) \mapsto \mathcal{D}(Fc, d)$$

$$\mathcal{C}^{op} \times \mathcal{D} \to \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Set}$$

$$(c, d) \mapsto (c, Ud) \mapsto \mathcal{C}(c, Ud)$$

$$\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{D} \arrow[r, bend left, "{\mathcal{D}(F-,=)}"] \arrow[r, bend right, "{\mathcal{C}(-,U=)}"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathrm{Set}
\end{tikzcd}$$

$f : c' \to c$

$g : d \to d'$

$$\begin{tikzcd}
\mathcal{D}(Fc, d) \arrow[r, "{\mathcal{D}(Ff,g)}"] \arrow[d, "\alpha_{c,d}"'] & \mathcal{D}(Fc', d') \arrow[d, "\alpha_{c',d'}"] \\
\mathcal{C}(c, Ud) \arrow[r, "{\mathcal{C}(f,Ug)}"'] & \mathcal{C}(c', Ud')
\end{tikzcd}$$

"taking the adjunct"

$$\begin{tikzcd}
u \arrow[r, mapsto] \arrow[d, "\wr"] & g \circ u \circ Ff \arrow[d, "\wr"] \\
\bar{u} \arrow[r, mapsto] & \overline{g \circ u \circ Ff} = Ug \circ \bar{u} \circ f
\end{tikzcd}$$

* the hom-isomorphism is useful for categories, but there are <u>many</u> other adjunctions with other kinds of abstract objects.

<u>adjunctions via units + counits</u>

recall $i : \{1\} \to U\mathbb{Z}$ coming from

$$\begin{array}{ccc} \mathrm{Grp}(\mathbb{Z}, \mathbb{Z}) & \xrightarrow{\sim} & \mathrm{Set}(\{1\}, U\mathbb{Z}) \\ id_{\mathbb{Z}} & \longmapsto & i = \overline{id_{\mathbb{Z}}} \end{array} \qquad (\mathbb{Z} = F(\{1\}))$$

this was "secretly" the unit of the adjunction $\mathrm{Grp} \underset{\perp}{\rightleftarrows} \mathrm{Set}$ on $\{1\} \in \mathrm{Set}$.

<u>Thm</u> given functors $F \dashv U : C \rightleftarrows D$ specifying an adjunction is equivalent to giving natural transformations $\eta : 1_C \Rightarrow UF$, $\varepsilon : FU \Rightarrow 1_D$

$$\begin{tikzcd}
U \arrow[r, Rightarrow, "\eta U"] \arrow[dr, Rightarrow, "id_U"'] & UFU \arrow[d, Rightarrow, "U\varepsilon"] \\
& U
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "id_F"'] & FUF \arrow[d, Rightarrow, "\varepsilon F"] \\
& F
\end{tikzcd}$$

Componentwise

$$\begin{tikzcd}
Ud \arrow[r, "\eta_{Ud}"] \arrow[dr, "1_{Ud}"'] & UFUd \arrow[d, "U\varepsilon_d"] \\
& Ud
\end{tikzcd}
\qquad
\begin{tikzcd}
Fc \arrow[r] \arrow[dr] & FUFc \arrow[d] \\
& Fc
\end{tikzcd}$$

$\Rightarrow$: $\alpha_{cd} : D(Fc, d) \xrightarrow{\sim} C(c, Ud)$

<u>construct unit</u>
- components
- naturality

$\alpha_{c,Fc} : D(Fc, Fc) \to C(c, UFc)$

$1_{Fc} \mapsto \overline{1_{Fc}} =: \eta_c$

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "\eta_c"'] \arrow[dr, phantom, "?"] & c' \arrow[d, "\eta_{c'}"] \\ UFc \arrow[r, "UFf"'] & UFc' \end{tikzcd}$$

verify that adjuncts are equal, then use bijectivity of α

$$\frac{c \xrightarrow{f} c' \xrightarrow{\eta_{c'}} UFc'}{Ff \circ id_{Fc'} = Ff}$$

$$\frac{c \xrightarrow{\eta_c} UFc \xrightarrow{UFf} UFc'}{1_{Fc} \circ Ff = Ff}$$

....

<u>Q&A</u>: Michael – self-adjoint functors?

$* : Vect_k \to Vect_k$

Josh – counterexample to $\eta \Delta id \Leftrightarrow \varepsilon \Delta id$?

not sure!

Me – favorite adjunction?

$Gph \rightleftarrows Cat$

special properties?

idempotence