

10/3 Christina Vasilakopoulos
 (John is in UK today + Friday)

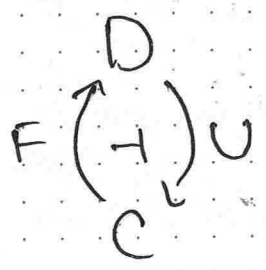
will need 2-categories for homotopy

\mathcal{C} (locally small) category

$$\text{hom}_{\mathcal{C}} = \mathcal{C}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Set}$$

$$(x, y) \mapsto \mathcal{C}(x, y)$$

$$\begin{array}{ccc} f: x' \rightarrow x & & \\ g: y \rightarrow y' & & \\ (f, g) \downarrow & & \downarrow g \circ - \circ f \\ (x', y') \mapsto \mathcal{C}(x', y') & & \end{array}$$



$$\alpha_{c,d}: \mathcal{D}(Fc, d) \xrightarrow{\sim} \mathcal{C}(c, Ud)$$

$$\begin{array}{ccccc} \mathcal{C}^{\text{op}} \times \mathcal{D} & \xrightarrow{F^{\text{op}} \times 1} & \mathcal{D}^{\text{op}} \times \mathcal{D} & \xrightarrow{\mathcal{D}(-, -)} & \text{Set} \\ (c, d) & \mapsto & (Fc, d) & \mapsto & \mathcal{D}(Fc, d) \end{array}$$

$$\begin{array}{ccccc} \mathcal{C}^{\text{op}} \times \mathcal{D} & \longrightarrow & \mathcal{C}^{\text{op}} \times \mathcal{C} & \longrightarrow & \text{Set} \\ (c, d) & \mapsto & (c, Ud) & \mapsto & \mathcal{C}(c, Ud) \end{array}$$

$$\begin{array}{ccc} \mathcal{C}^{\text{op}} \times \mathcal{D} & \xrightarrow{\mathcal{D}(F-, -)} & \text{Set} \\ \parallel \alpha & & \\ \mathcal{C}^{\text{op}} \times \mathcal{C} & \xrightarrow{\mathcal{C}(-, U-)} & \text{Set} \end{array}$$

$$\begin{array}{ccc} f: c \rightarrow c' & \mathcal{D}(Fc, d) \xrightarrow{\alpha(Ff, g)} \mathcal{D}(Fc', d') & \text{"taking the adjoint"} \\ g: d \rightarrow d' & \alpha_{c,d} \downarrow & \downarrow \alpha_{c',d'} \\ & \mathcal{C}(c, Ud) \xrightarrow{\alpha(f, U_g)} \mathcal{C}(c', Ud') & \end{array}$$

$$u \mapsto g \circ u \circ Ff$$

$$\downarrow$$

$$\downarrow$$

$$\bar{u} \mapsto \overline{g \circ u \circ Ff} = Ug \circ \bar{u} \circ F$$

* the hom-isomorphism is useful for categories, but there are many other adjunctions with other kinds of abstract objects.

adjunctions via units + counits

recall $i: \{1\} \rightarrow U\mathbb{Z}$ coming from

$$\text{Grp}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\sim} \text{Set}(\{1\}, U\mathbb{Z}) \quad (\mathbb{Z} = F(\{1\}))$$

$$\text{id}_{\mathbb{Z}} \mapsto i = \overline{\text{id}_{\mathbb{Z}}}$$

this was "secretly" the unit of the adjunction $\text{Grp} \overset{\perp}{\rightleftarrows} \text{Set}$ on $\{1\} \in \text{Set}$.

Thm given functors $F \dashv U: C \rightleftarrows D$ specifying an adjunction is equivalent to giving natural transformations $\eta: 1_C \Rightarrow UF$, $\epsilon: FU \Rightarrow 1_D$

$$\begin{array}{ccc} U & \xrightarrow{\eta_U} & UFU \\ & \searrow \text{id}_U & \downarrow U\epsilon \\ & & U \end{array}$$

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FUF \\ & \searrow \text{id}_F & \downarrow F\epsilon \\ & & F \end{array}$$

Componentwise

$$\begin{array}{ccc} U_d & \xrightarrow{\eta_{U_d}} & UFU_d \\ & \searrow \text{id}_{U_d} & \downarrow U\epsilon_d \\ & & U_d \end{array}$$

$$\begin{array}{ccc} F_c & \xrightarrow{F\eta_c} & FUF_c \\ & \searrow & \downarrow F\epsilon_c \\ & & F_c \end{array}$$

$$\Rightarrow: \alpha_{cd}: D(Fc, d) \xrightarrow{\sim} C(c, Ud)$$

construct unit

- components
- naturality

$$\alpha_{c, Fc}: D(Fc, Fc) \rightarrow C(c, UFc)$$

$$1_{Fc} \mapsto \overline{1_{Fc}} =: \eta_c$$

$$\begin{array}{ccc} c & \xrightarrow{f} & c' \\ \eta_c \downarrow & ? & \downarrow \eta_{c'} \\ UFc & \xrightarrow{UFf} & UFc' \end{array}$$

verify that adjoints are equal,
then use bijectivity of α

$$\begin{array}{ccc} c & \xrightarrow{\eta_c} & UFc \xrightarrow{UFf} UFc' \\ \downarrow 1_{Fc} & & \\ \eta_c \circ Ff & = & Ff \end{array}$$

$$\begin{array}{ccc} c & \xrightarrow{f} & c' \xrightarrow{\eta_{c'}} UFc' \\ \hline & & \downarrow id_{c'} \\ & & Ff \circ id_{c'} = Ff \end{array}$$

....

Q&A: Michael - self-adjoint functors?

$$*: Vect_k \rightarrow Vect_k$$

Josh - counterexample to $\eta \Delta id$
not sure! $\varepsilon \Delta id$?

Me - favorite adjunction?

$$Gph \rightleftharpoons Cat$$

special properties?

idempotence