2-Category Theory (Christina) 10/5

we know that functors compose

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[rr, bend right, "G \circ F"'] & D \arrow[r, "G"] & E \end{tikzcd}$$

but natural transformations have two ways of composing:

$$\begin{tikzcd} C \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] & D \end{tikzcd}$$

<u>vertical</u>:

$(\beta \cdot \alpha)_c = Fc \xrightarrow{\alpha_c} Gc \xrightarrow{\beta_c} Hc$

$$\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & E \end{tikzcd} =: \begin{tikzcd} C \arrow[r, bend left, "H \circ F"] \arrow[r, bend right, "K \circ G"'] \arrow[r, phantom, "\Downarrow\beta\circ\alpha"] & E \end{tikzcd}$$

<u>horizontal</u>:

$(\beta \circ \alpha)_c$:

$$\begin{tikzcd} & K(F(c)) \arrow[dr, "K\alpha_c"] & \\ H(F(c)) \arrow[ur, "\beta_{Fc}"] \arrow[rr, dashed] \arrow[dr, "H\alpha_c"'] & & K(G(c)) \\ & H(G(c)) \arrow[ur, "\beta_{Gc}"'] & \end{tikzcd}$$

(naturality of β makes this commute)

we can draw (and compute!) general "pasting diagram"

- whiskering

$$\xrightarrow{F} \Downarrow\alpha \xrightarrow{G} := \begin{tikzcd} \cdot \arrow[r, bend left, "F"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow id"] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\alpha"] & \cdot \arrow[r, bend left, "G"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow id"] & \cdot \end{tikzcd}$$

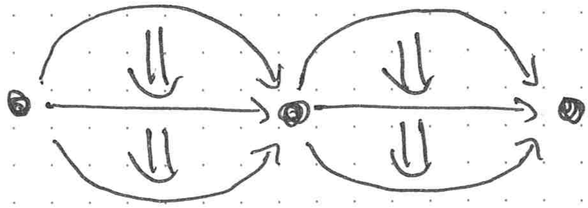
$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "H"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \cdot \arrow[dr, "G"] \arrow[d, phantom, "\Downarrow\beta"] & \\ \cdot \arrow[ur, "L"] \arrow[rr, "K"'] & & B \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[rr, bend left, "F"] \arrow[r, "H"'] \arrow[rr, phantom, "\Downarrow\alpha"] & \cdot \arrow[r, "L"'] & \cdot \arrow[r, "G"] & B \end{tikzcd}$$

$$\bullet$$

$$\begin{tikzcd} A \arrow[r, "H"] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\beta"] & B \end{tikzcd}$$

choice of order of pasting does <u>not</u> matter — interchange law

interchange law



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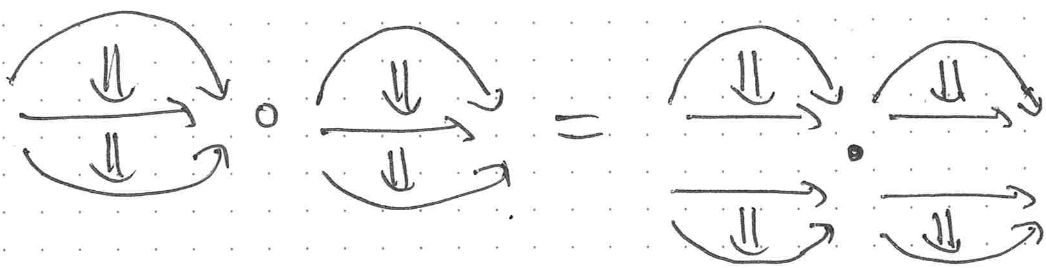

definition:

a <u>2-category</u> K is given by:

- objects $x, y, \ldots$ (0-cells)
- $\forall x, y$ a category $K(x,y)$ — objects $x \longrightarrow y$ (1-cells)

  morphisms $\begin{tikzcd} x \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & y \end{tikzcd}$ (2-cells)

  composition in $K(x,y)$ is vertical composition
- $\forall x, y, z$ a functor

  $\circ : K(y,z) \times K(x,y) \longrightarrow K(x,z)$

horizontal composition $\left(\begin{tikzcd} y \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\beta"] & z \end{tikzcd}, \begin{tikzcd} x \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd}\right) \mapsto \begin{tikzcd} x \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\beta\circ\alpha"] & z \end{tikzcd}$

(plus coherence)

examples

- Cat – categories^0, functors^1, natural trans^2
- Top – spaces, maps, homotopies

(2-categories form a 3-category)

definition:

an <u>adjunction</u> in a 2-category K is:

- 0-cells x and y
- 1-cells $\begin{tikzcd} x \arrow[r, bend left, "f"] & y \arrow[l, bend left, "g"] \end{tikzcd}$
- 2-cells $\begin{tikzcd} x \arrow[rr, bend left, "id_x"] \arrow[dr, "f"'] \arrow[rr, phantom, "\Downarrow\eta"] & & x \\ & y \arrow[ur, "g"'] & \end{tikzcd}$ $\begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[ur, "g"] \arrow[rr, bend right, "id_y"'] \arrow[rr, phantom, "\Downarrow\varepsilon"] & & y \end{tikzcd}$

such that: triangle identities

$$\begin{tikzcd} g \arrow[r, Rightarrow, "\eta\circ 1_g"] \arrow[dr, Rightarrow, "1_g"'] & gfg \arrow[d, Rightarrow, "1_g\circ\varepsilon"] \\ & g \end{tikzcd} \qquad \begin{tikzcd} f \arrow[r, Rightarrow, "1_f\circ\eta"] \arrow[dr, Rightarrow, "1_f"'] & fgf \arrow[d, Rightarrow, "\varepsilon\circ 1_f"] \\ & f \end{tikzcd}$$

looks better with pasting diagrams!

$$\begin{tikzcd} & x \arrow[d, equal] \arrow[drr, bend left] & & \\ y \arrow[ur, bend left] \arrow[r] \arrow[drr, bend right] \arrow[r, phantom, shift left=3, "\Downarrow 1_g"] & x \arrow[r] \arrow[dr, phantom, "\Downarrow\varepsilon"] & y \arrow[r] \arrow[d, equal] \arrow[u, phantom, "\Downarrow\eta"] & x \\ & & y \arrow[ur, bend right] \arrow[ur, phantom, "\Downarrow 1_g"] & \end{tikzcd} = \begin{tikzcd} & x \arrow[d] \arrow[dr, bend left] & \\ y \arrow[ur, bend left] \arrow[dr, bend right] \arrow[r, phantom, "\Downarrow\varepsilon"] & {} \arrow[r, phantom, "\Downarrow\eta"] & x \\ & y \arrow[ur, bend right] & \end{tikzcd} = 1_g$$