2-categories & String Diagrams

formal definition: a <u>2-category</u> X is

- a collection of objects a, b (0-cells)
- $\forall a,b \in X$ a hom-category $X(a,b)$

$$\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \end{tikzcd}$$

- objects f, g (1-cells)
- morphisms α, β (2-cells)
- composition "$\bullet$" – vertical

- $\forall a,b,c \in X$ a functor

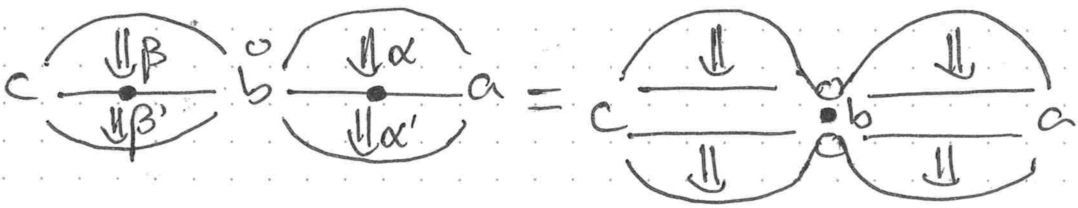
$$\circ : X(b,c) \times X(a,b) \longrightarrow X(a,c) \quad \text{(1-cell composition)}$$

on objects: 1-cell composition

on morphisms: 2-cell horizontal composition

functoriality ≈ interchange law

$$(\beta' \bullet \beta) \circ (\alpha' \bullet \alpha) = (\beta' \circ \alpha') \bullet (\beta \circ \alpha)$$



such that $\circ$ is <u>associative</u>

and <u>unital</u>: a functor $1_a : \mathbb{1} \to X(a,a)$

where $\mathbb{1}$ is the terminal category

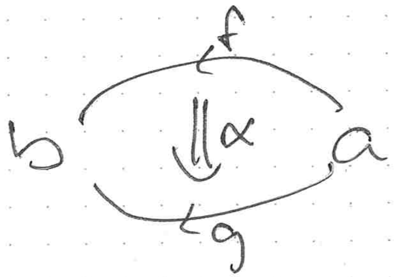
(picks out the identity 2-cell of 1_a)

$$\begin{tikzcd}
X(c,d)\times X(b,c)\times X(a,b) \arrow[r, "\circ\times id"] \arrow[d, "id\times\circ"'] \arrow[dr, phantom, "\text{associativity}"] & X(b,d)\times X(a,b) \arrow[d, "\circ"] \\
X(c,d)\times X(a,c) \arrow[r, "\circ"'] & X(a,d)
\end{tikzcd}$$

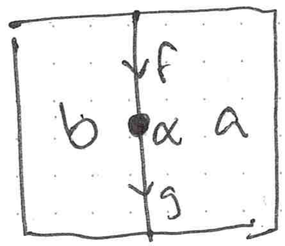
$$\begin{tikzcd}
1\times X(a,b) \arrow[d, "u_a\times id"'] \arrow[dr, phantom, "\text{left unit}"] & X(a,b) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "id"] \arrow[dr, phantom, "\text{right unit}"] & X(a,b)\times 1 \arrow[d, "id\times u_b"] \\
X(a,a)\times X(a,b) \arrow[r, "\circ"'] & X(a,b) & X(a,b)\times X(a,a) \arrow[l, "\circ"]
\end{tikzcd}$$

(these are all functors)

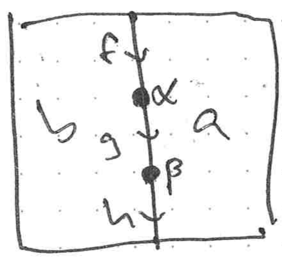
we can draw 2-morphisms in a 2-category in another way: string diagrams.

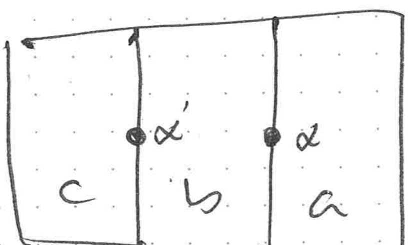


Poincaré duality



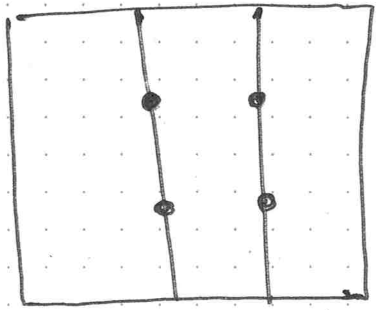
vertical composition $\Longrightarrow$





$\Longleftarrow$ horizontal composition

interchange :



<u>theorem</u> given two adjunctions

$$\begin{tikzcd} D \arrow[rr, phantom, "\perp"] \arrow[rr, bend right, "U"'] & & C \arrow[ll, bend right, "F"'] \end{tikzcd}$$

$\eta : 1_C \Rightarrow UF$

$\varepsilon : FU \Rightarrow 1_D$ (△ ids)

$$\begin{tikzcd} E \arrow[rr, phantom, "\perp"] \arrow[rr, bend right, "U'"'] & & D \arrow[ll, bend right, "F'"'] \end{tikzcd}$$

η', ε'

we get <u>another</u> adjunction

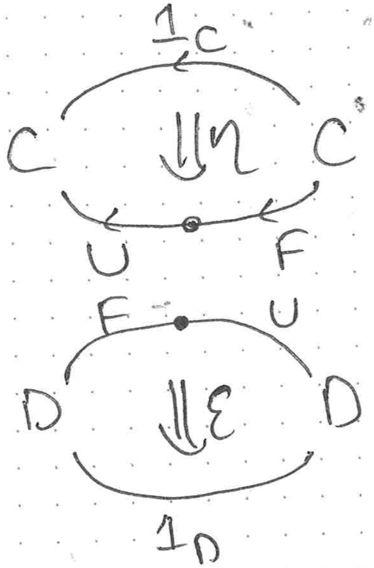
$$\begin{tikzcd} E \arrow[rr, phantom, "\perp"] \arrow[rr, bend right, "UU'"'] & & C \arrow[ll, bend right, "F'F"'] \end{tikzcd}$$

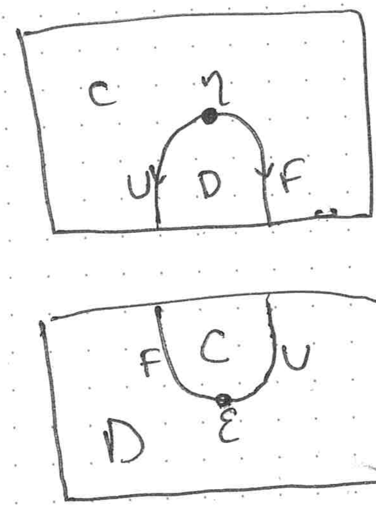
with certain unit + counit

<u>proof</u> in Cat, just compose that natural hom-isomorphisms :

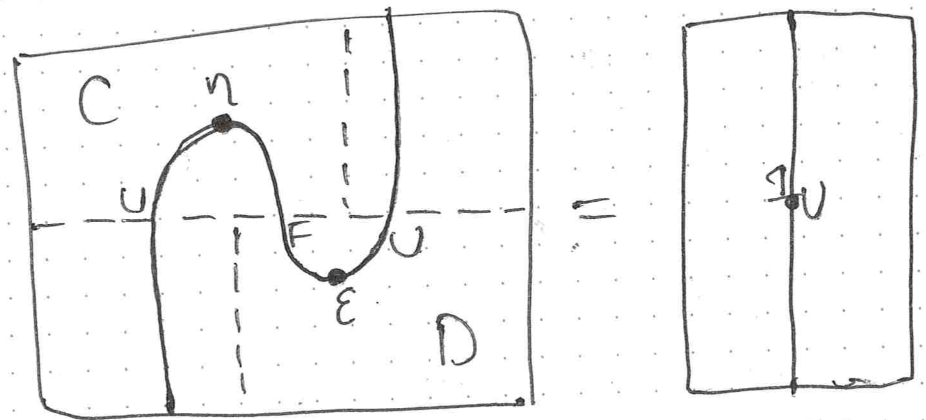
$$E(F'Fc, e) \xrightarrow{\sim} D(Fc, U'e) \xrightarrow{\sim} C(c, UU'e)$$

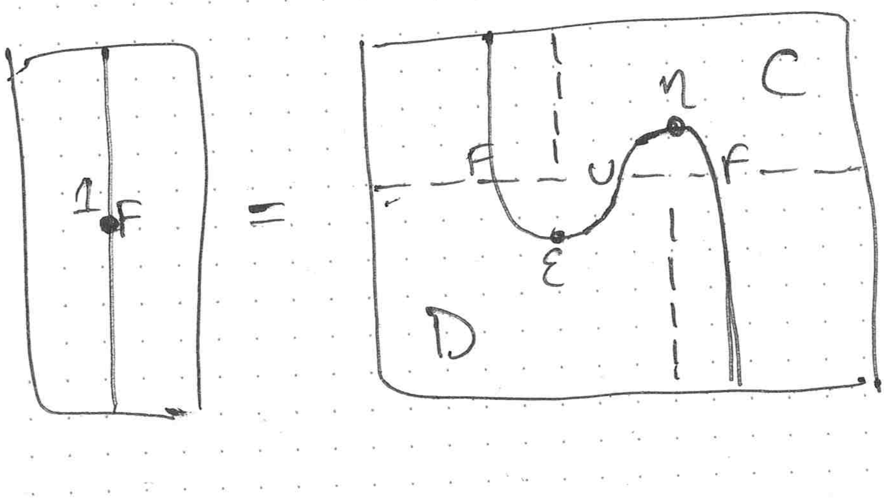
but in a general 2-category, need to define unit + counit



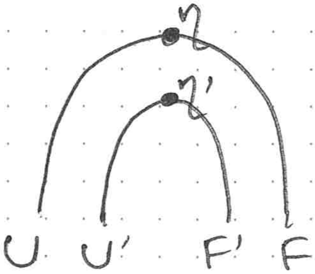


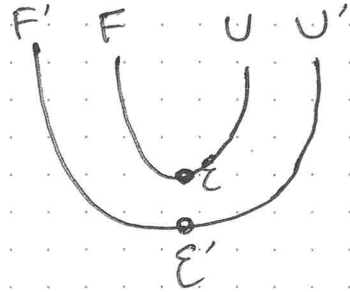
triangle identities become
<u>zigzag identities</u>:





the unit & counit we want are:





(use interchange & zigzag id's
to prove "thickened" zigzag ids]