The "main spine" of mathematics

The category Cat

$$\begin{tikzcd}
 & \mathrm{Set} \arrow[dl, shift right] \arrow[dr, shift left] & \\
\mathrm{Mon} \arrow[ur, shift right] \arrow[rr, shift left] \arrow[d, shift right] & & \mathrm{Grp} \arrow[ul, shift left] \arrow[ll, shift left] \arrow[d, shift right] \\
\mathrm{ComMon} \arrow[u, shift right] \arrow[rr, shift left] \arrow[d, shift right] & & \mathrm{AbGrp} \arrow[u, shift right] \arrow[ll, shift left] \arrow[d, shift right] \\
\mathrm{Rig} \arrow[u, shift right] \arrow[rr, shift left] \arrow[d, shift right] & & \mathrm{Ring} \arrow[u, shift right] \arrow[ll, shift left] \arrow[d, shift right] \\
\mathrm{ComRig} \arrow[u, shift right] \arrow[rr, shift left] & & \mathrm{ComRing} \arrow[u, shift right] \arrow[ll, shift left]
\end{tikzcd}$$

e.g. every group has an underlying monoid UG; every monoid has a free group FM, the group with generators $m \in M$ and relations $m_1 m_2 = m_3$ whenever this holds in M.

$$\begin{tikzcd}
\mathrm{Mon} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Grp} \arrow[l, shift left]
\end{tikzcd}$$

<u>Puzzle</u>: check $\mathrm{Grp}(FM, G) \cong \mathrm{Mon}(M, UG)$

we get good stuff by applying the left adjoints to a simple set: ⟶

$$\begin{tikzcd}
 & \emptyset \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
\{0\} \arrow[d, mapsto] & & \{0\} \arrow[d, mapsto] \\
\{0\} \arrow[d, mapsto] & & \{0\} \arrow[d, mapsto] \\
\mathbb{N} \arrow[d, mapsto] & & \mathbb{Z} \arrow[d, mapsto] \\
\mathbb{N} & & \mathbb{Z}
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \{x\} \arrow[dl, mapsto] \arrow[dr, mapsto] & & \\
\{0, x, x+x, \ldots\} \arrow[r, phantom, "\cong"] & \mathbb{N} \arrow[d, mapsto] & & \mathbb{Z} \arrow[d, mapsto] \arrow[r, phantom, "\cong"] & \{\ldots, -x, 0, x, \ldots\} \\
 & \mathbb{N} \arrow[d, mapsto] & & \mathbb{Z} \arrow[d, mapsto] & \\
 & \mathbb{N}[x] \arrow[d, mapsto] & & \mathbb{Z}[x] \arrow[d, mapsto] & \\
 & \mathbb{N}[x] & & \mathbb{Z}[x] &
\end{tikzcd}$$

<u>Puzzle</u>: $\{x, y\}$

Monads & Comonads

suppose we have an adjunction in any 2-category X

$$\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & D \arrow[l, bend left, "U"] \end{tikzcd}$$

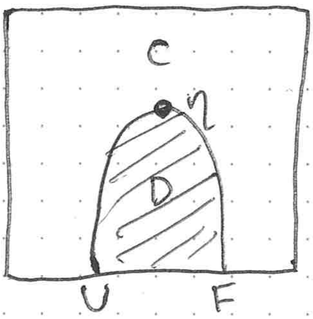
$\eta : 1_C \Rightarrow UF$
$\varepsilon : FU \Rightarrow 1_D$ ($\triangle$ ids)

(e.g. Set $\rightleftarrows$ Mon)

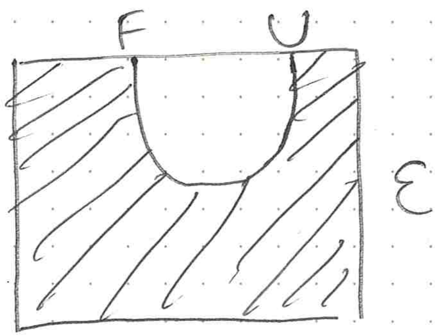
$X(C,C)$ is a category

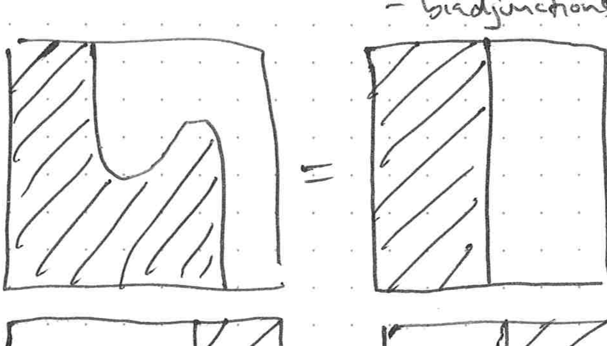
objects: 1-cells $C \to C$ (monoid generated by UF) $1_C, UF, UFUF, \ldots$

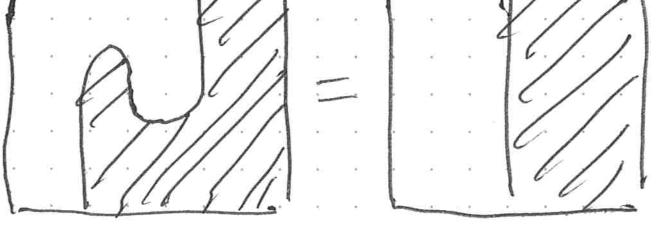
morphisms: 2-cells $\Downarrow$ $\eta, U\varepsilon F, U\varepsilon\varepsilon F, \ldots$



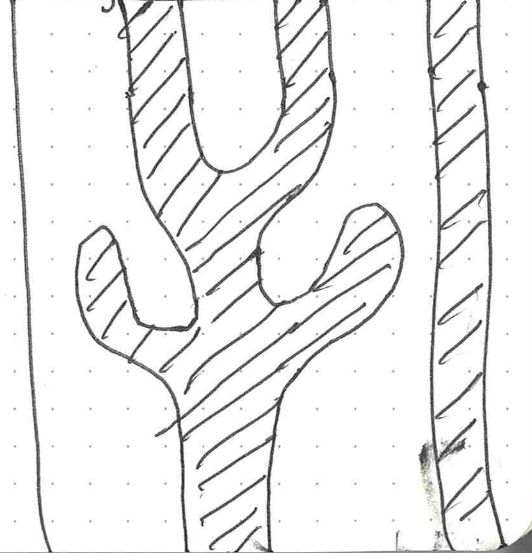
dark stripes can merge using counit and stop using unit (can't split or stop - biadjunctions)



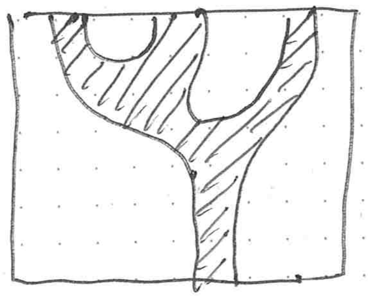




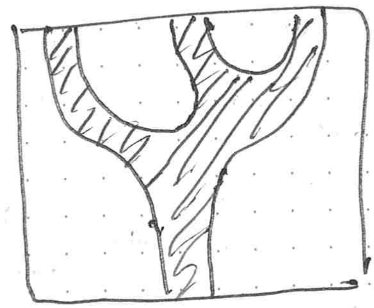
typical morphism in $X(C,C)$

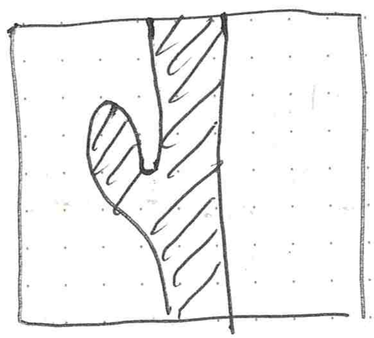


next time we'll see these diagrams obey some relations:

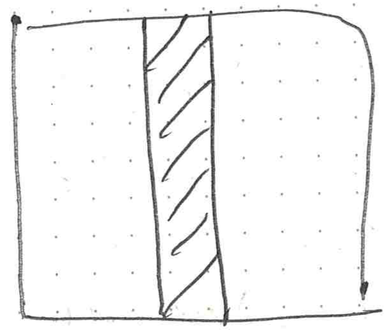


= assoc





= unit



and vice versa

will see how this category is fundamental to homotopy, and all math (every adjunction gives a cohomology theory)

Q&A: isotopy of diagrams