Monads

defn: if $X$ is a 2-category, a monad in $X$ consists of
1. an object $x \in X$
2. a morphism $T: x \to x$
3. two 2-morphisms
   - multiplication $\mu: T \circ T \Rightarrow T$
   - unit $\eta: 1_x \Rightarrow T$
4. with properties
   - associativity
     \[ T^3 \xrightarrow{T} T^2 \xrightarrow{\mu} T \]
     \[ T^2 \xrightarrow{\mu} T \]
   - unitality
     \[ 1_x \circ T = T = T \circ 1_x \]
     \[ 2 \circ T \xrightarrow{1_T} T \circ T \xrightarrow{\mu} T \]
     \[ T \circ T \xrightarrow{\mu} T \]
Thm: if $X$ is a 2-category containing an adjunction
\[
\begin{array}{ccc}
C & \xrightarrow{F} & D \\
\downarrow & & \downarrow \\
U & \xleftarrow{U} & \text{id}
\end{array}
\]
then we get a monad,
1. $C$ as the object
2. $T = U \circ F$
3. $\mu = U \circ E \circ F \circ F: UFUF \to UF$
$\eta = \eta$

Proof:
\[
\begin{array}{ccc}
\text{associativity} \\
\alpha & = & \beta
\end{array}
\]
by,

in general
\[
\begin{array}{ccc}
\alpha & = & \beta
\end{array}
\]

interchange law
\[
\begin{array}{ccc}
\alpha & \quad & \beta \\
\frac{1}{\alpha} & = & \text{law}
\end{array}
\]
and left unit law by

implies

given an adjunction, we've seen that a general morphism in the category \text{hom}(C,C) looks like:

or more simply:
Then the sadistic tree-trimmers come to town:

(normal form shortens the branches)

Use notation which implies associativity:

What is this?

an order-preserving function of finite linear orders

we're seeing a model of $\Delta a$ in $\text{hom}(C,C)$

(no crossings $\nabla$)
define the augmented simplex category \( \Delta a \) has:

- finite ordinals as objects
  
  \[
  [0] = \{0\} \\
  [1] = \{0, 1\} \\
  [2] = \{0, 1, 2\}
  \]

- order-preserving maps as morphisms

\( \Delta \), the simplex category, is the full subcategory of \( \Delta a \) containing only nonempty ordinals \([0], [2], [3], \ldots\)

and all order-preserving maps

Q&A: a "copy" of \( \Delta a \) means a functor

\[ \Delta a \to \text{hom}(C, C) \]

(there is "the walking adjunction")

\( \text{Adj} X \), \( X \): 2-functor \( \text{Adj} \to X \)

- interesting adj's besides \( \text{Cat} \)?
  (think of monoidal)